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CLASSIFICATION OF CUBIC DIFFERENTIAL SYSTEMS WITH A MONODROMIC CRITICAL POINT AND MULTIPLE LINE AT INFINITY

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Abstract. *In this article, we classified the cubic differential systems with a non-degenerate monodromic critical point and multiple line at infinity. We show that there are 5 distinct classes (respectively, 10, 6, 6) of such systems wich have the line at infinity of multiplicity 2 (respectively, 3, 4, 5).*

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I. Introduction

We consider diferential cubic systems of the form

$$\dot{x}=y+ax^2+cxy+fy^2+kx^3+mx^2y+pxy^2+ry^3\equiv P(x,y),$$
$$\dot{y}=-(x+gx^2+dxy+by^2+sx^3+qx^2y+nxy^2+ly^3)\equiv Q(x,y),\ \gcd(P,Q)=1 \tag{1}$$

and the vector fields $\mathbb{X}=P(x,y)\frac{\partial}{\partial x}+Q(x,y)\frac{\partial}{\partial y}$ assosiated to systems (1).

For (1) the origin $O(0,0)$ is a non-degenerate critical point of a center or a focus type, i.e. is monodromic.

Denote $k(x,y)=sx^4+(k+q)x^3y+(m+n)x^2y^2+(l+p)xy^3+ry^4$. If $k(x,y)\equiv 0,$ then the infinity is degenerate for (1), i.e. it consists only from critical points. In this work we suppose that $k(x,y)\not\equiv 0.$

Let $\bar{P}(x,y,Z),\bar{Q}(x,y,Z)$ are the homogenization of the polynomials $P(x,y),Q(x,y)$ respectively, i.e.

$$\bar{P}(x,y,Z)=yZ^2+(ax^2+cxy+fy^2)Z+kx^3+mx^2y+pxy^2+ry^3,$$
$$\bar{Q}(x,y,Z)=-(xZ^2+(gx^2+dxy+by^2)Z+sx^3+qx^2y+nxy^2+ly^3),$$

and $\bar{\mathbb{X}}=\bar{P}(x,y,Z)\frac{\partial}{\partial x}+\bar{Q}(x,y,Z)\frac{\partial}{\partial y}$. Assume that the line at infinity $Z=0$ is not full of critical points. We say that $Z=0$ has multiplicity $m+1$ if m is the greatest positive integer such that Z^m divides $\mathbb{E}(\bar{\mathbb{X}})=\bar{P}\cdot\bar{\mathbb{X}}(\bar{Q})+\bar{Q}\cdot\bar{\mathbb{X}}(\bar{P})$. The polynomial $\mathbb{E}(\bar{\mathbb{X}})$ has the form

$$\mathbb{E}(\bar{\mathbb{X}})=C_0(x,y)+C_1(x,y)Z+\cdots+C_8(x,y)Z^8,$$

where $C_j(x,y), j=\overline{1,8}$ are polynomials in x and y. If $C_j(x,y)\equiv 0, j=\overline{0,m}$, then $m+2$ is the multiplicity of the line at infinity.

II. Cubic systems (1) with the line at infinity of multiplicity two

Theorem 1. *The line at infinity $Z=0$ has the multiplicity at least two for system (1) if and only if one of the following five sets of conditions holds:*

$$n = k = m = s = p = q = 0,\ r \neq 0; \tag{2z1}$$
$$k = m = p = r = 0; \tag{2z2}$$
$$k = m = q = s = 0, l = nr/p; \tag{2z3}$$
$$k = s = 0, l = qr/m, n = pq/m; \tag{2z4}$$
$$l = rs/k,\ n = ps/k,\ q = ms/k. \tag{2z5}$$

Proof. We have $C_0(x,y) = k(x,y)C_{01}(x,y)$, where $C_{01}(x,y) = (kq - ms)x^4 + 2(kn - p\,s)x^3y + (3kl + mn - pq - 3rs)x^2y^2 + 2(lm - qr)xy^3 + (lp - nr)y^4$.
It is easy to show that the identity $C_{01}(x,y) \equiv 0$ gives us the sets of conditions (2z1)-(2z5).

III. Cubic systems (1) with the line at infinity of multiplicity three

Remark 1. *The transformation* $x \to y, y \to x, t \to -t$ *preserve the form of the system* (1) *and change the coefficients in the following way:* $a \leftrightarrow b, c \leftrightarrow d, \dots, r \leftrightarrow s$.

Let $\{\mathcal{F}(a,b,c,d,\dots,r,s)\}$ be a set of conditions. Denote

$$\{\mathcal{F}(a,b,c,d,\dots,r,s)\}^{\leftrightarrow} = \{\mathcal{F}(b,a,d,c,\dots,s,r)\}.$$

Theorem 2. *Modulo the transformations* $x \to y, y \to x, t \to -t$, *the line at infinity* $Z = 0$ *has the multiplicity at least three for system (1) if and only if one of the following ten sets of conditions holds:*

$$a = c = f = k = m = p = r = 0; \tag{3z1}$$

$$c = aq/s,\ f = an/s, k = l = m = p = r = 0; \tag{3z2}$$

$$a = 0, c = fq/n, k = l = m = p = r = s = 0; \tag{3z3}$$

$$a = f = k = l = m = n = p = r = s = 0,\ q \neq 0; \tag{3z4}$$

$$a = gp/n, c = dp/n, k = 0, l = p, m = q = 0, r = p\hat{}2/n, s = 0; \tag{3z5}$$

$$b = fq/m, d = cq/m, g = aq/m, k = 0, l = qr/m, n = pq/m, s = 0,\ q \neq 0; \tag{3z6}$$

$$d = \frac{bm^3 - fm^2q + cq^2r}{mqr}, g = \frac{aq}{m}, k = 0, l = \frac{qr}{m}, n = m + \frac{q^2r}{m^2}, p = \frac{m^3 + q^3r}{mq}, s = 0; \tag{3z7}$$

$$b = fq/m, g = aq/m, k = l = 0, n = m, p = m\hat{}2/q, r = s = 0; \tag{3z8}$$

$$b = fs/k, d = cs/k, g = as/k, l = rs/k, n = ps/k, q = ms/k; \tag{3z9}$$

$$b = \frac{fk^2s + gkrs - ars^2}{k^3},\ d = \frac{s(ck^3 + gk^2p - akps - gkrs + ars^2)}{k^4},$$
$$l = \frac{rs}{k},\ m = \frac{k^4 + kps^2 - rs^3}{k^2s},\ n = \frac{ps}{k},\ q = (k^4 + kps^2 - rs^3)/k^3. \tag{3z10}$$

Proof. In each of the conditions $\{2z1, k(x,y) \not\equiv 0\} - \{2z5, k(x,y) \not\equiv 0\}$ we will solve the identity $C_1(x,y) \equiv 0$.

Conditions (2z1). Under these conditions the polynomial $C_1(x,y)$ has the form $C_1(x,y) = y^5[3l(al - gr)x^2 + 2(cl^2 + alr - dlr - gr^2)xy + (fl^2 - blr + clr - dr^2)y^2]$. Taking into account that $k(x,y) = -y^3(lx - ry) \not\equiv 0$, the identity $C_1(x,y) \equiv 0$ yield:

$$f = br/l,\ a = gr/l,\ c = dr/l; \tag{2}$$

$$d = g = l = 0. \tag{3}$$

Using transformations $x \to y, y \to x, t \to -t$, we obtain that the conditions $\{2z1, (2)\}$ (respectively, $\{2z1, (3)\}$) are contained in $(3z9)$ (respectively, in (3z2)).

Conditions (2z2). In this case we have $k(x,y) = -x(sx^3 + qx^2y + nxy^2 + ly^3)$ and $C_1(x,y) = -\frac{k(x,y)}{x}[(aq - cs)x^4 + 2(an - fs)x^3y + (3al + cn - fq)x^2y^2 + 2clx\,y^3 + fly^4]$.

The identity $C_1(x,y) \equiv 0$ is realisable if one of the following four sets of conditions holds:

$$a = c = f = 0; \tag{4}$$

$$c = aq/s, f = an/s, l = 0; \tag{5}$$

$$a = 0, c = fq/n, l = s = 0; \tag{6}$$

$$a = f = l = n = s = 0, q \neq 0. \tag{7}$$

The conditions (4) – (7), together with (2z2), lead us to conditions (3z1)-(3z4), respectively.

Conditions (2z3). The polynomials $k(x,y)$ and $C_1(x,y)$ look as:

$k(x,y) = y^2(py - nx)(px + ry)/p$, $C_1(x,y) = -y^3(px + ry)C_{11}(x,y)/p^2$,

where

$C_{11}(x,y) = 2np(gp - an)x^3 - (cn^2p + anp^2 - dnp^2 - gp^3 + 3an^2r - 3gnpr)x^2y - 2r(cn^2 + anp - dnp - gp^2)xy^2 + (fnp^2 - bp^3 - fn^2r + bnpr - cnpr + dp^2r)y^3$, and $C_{11}(x,y) \equiv 0 \Rightarrow$

$$a = gp/n, c = dp/n, r = p\char`^2/n; \tag{8}$$

$$b = dr/p, g = n = 0; \quad (9)$$

$$a = gp/n, c = dp/n, f = bp/n. \quad (10)$$

We have $\{(2z3),(8)\} \equiv (3z5)$, $\{(2z3),(9),r = 0\}^{\leftrightarrow} \subset (3z4)$, $\{(2z3),(9),r \neq 0\}^{\leftrightarrow} \subset (3z2)$, $\{(2z3),(10),l = 0\}^{\leftrightarrow} \subset (3z6)$ and $\{(2z3),(10),l \neq 0\}^{\leftrightarrow} \subset (3z9)$.

Conditions (2z4). In this case we find that

$k(x,y) = y(\mathrm{my} - \mathrm{qx})(\mathrm{m}x\hat{}2 + \mathrm{pxy} + \mathrm{ry}\hat{}2\,)/\mathrm{m}$

and

$C_1(x,y) = (k(x,y)\; C_{12}(x,y))/(m\hat{}2\;(my - qx)\,)$,

where

$C_{12}(x,y) = -\mathrm{mq}(\mathrm{gm} - \mathrm{aq})x^4 - 2\mathrm{pq}(\mathrm{gm} - \mathrm{aq})x^3\mathrm{y} + (\mathrm{d}m^3 - \mathrm{g}m^2\mathrm{p} + \mathrm{bm}^{2\mathrm{q}} - cm^2\mathrm{q} + \mathrm{ampq} - \mathrm{dmpq} - \mathrm{fm}q^2 + \mathrm{cp}q^2 - 3\mathrm{gmqr} + 3\mathrm{a}q^2\mathrm{r})x^2y^2 + 2(\mathrm{b}m^3 - \mathrm{f}m^2\mathrm{q} - \mathrm{g}m^2\mathrm{r} + \mathrm{amqr} - \mathrm{dmqr} + cq^2\mathrm{r})\mathrm{xy}^3 + (\mathrm{b}m^2\mathrm{p} - \mathrm{fmpq} - \mathrm{d}m^2r - \mathrm{bmqr} + \mathrm{cmqr} + \mathrm{fq}\hat{}2\mathrm{r})y^4$.

The identity $C_{13}(x,y) \equiv 0$ gives us the following six sets of conditions:

$$b = fq/m, d = cq/m, g = aq/m, q \neq 0; \quad (11)$$

$$d = cq/m + m(bm - fq)/qr, g = aq/m, p = m^2/q + q\,r/m\,; \quad (12)$$

$$b = fq/m, g = aq/m, p = m^2/q, r\; = 0; \quad (13)$$

$$d = bp/r, \mathrm{g} = \mathrm{bm}/\mathrm{r}, \mathrm{q} = 0; \quad (14)$$

$$b = 0, g = dm/p, q = r = 0; \quad (15)$$

$$b = d = p = q = r = 0. \quad (16)$$

It is easy to show that $\{(2z4,(j+5)\} \equiv (3zj), j = 6,7,8$ and $\{(2z4),(14)\}^{\leftrightarrow} \subset (3z2)$, $\{(2z4),(15),m = 0\}^{\leftrightarrow} \subset (3z4)$, $\{(2z4),(15),m \neq 0\}^{\leftrightarrow} \subset (3z3)$, $\{(2z4),(16)\}^{\leftrightarrow} \subset (3z3)$.

Conditions (2z5). Under this conditions the polynomials $k(x,y)$ and $C_1(x,y)$ look as:
$k(x,y) = (sx + ky)(\mathrm{k}x^3 + \mathrm{m}x^2\mathrm{y} + \mathrm{pxy}^2 + \mathrm{ry}^3)/\mathrm{k}$, $C_1(x,y) = \frac{k(x,y)}{k(sx+ky)} C_{13}(x,y)$,

where

$C_{13}(x,y) = (gk^3 - ak^2s + dk^2s - gkms - cks^2 + ams^2)x^4 + 2(dk^3 + bk^2s - ck^2s - gkps$
$-fks^2 + aps^2)x^3y + (3bk^3 + dk^2m - gk^2p - 3fk^2s + bkms - ckms + akps - dkps -$
$3gkrs - fms^2 + cps^2 + 3ars^2)x^2y^2 + 2(bk^2m - gk^2r - fkms + akrs - dkrs +$
$crs^2)xy^3 + (bk^2p - dk^2r - fkps - bkrs + ckrs + frs^2)y^4.$

If $k(x,y) \not\equiv 0$ and $C_{13}(x,y) \equiv 0$, then one of the following two sets of equalities holds:

$$b = fs/k, d = cs/k, g = as/k; \qquad (17)$$

$$d=(bk^2p-fkps-bkrs+ckrs+frk^2)/(k^2 r),$$
$$g=(bk^3-fk^2s+ars^2)/(krs), \quad m=(k^4+kps^2-rs^3)/(k^2s). \qquad (18)$$

The conditions (17) and (18) toghether with (2z5) give us the conditions (3z9) and (3z10), respectively.

IV. Cubic systems (1) with the line at infinity of multiplicity four

Theorem 3. *Modulo the transformations* $x \to y, y \to x, t \to -t$, *the line at infinity* $Z = 0$ *has the multiplicity at least four for system (1) if and only if one of the following six sets of conditions holds:*

$$a = 0, c = b, f = g = k = l = m = n = p = r = s = 0, q \neq 0; \qquad (4z1)$$

$$a = c = f = k = l = m = n = p = r = s = 0, \ q \neq 0; \qquad (4z2)$$

$$b = \frac{aq}{s}, c = \frac{aq}{s}, f = 0, d = \frac{agq + a^2s + s^2}{as}, k = l = m = n = p = r = 0; \qquad (4z3)$$

$$a=\frac{m^3}{q(bm-fq)}, \ c=\frac{m^4+b^2mq^2-bfq^3}{q^2(bm-fq)}, \ g=\frac{m^2}{bm-fq}, \ k=0, \ l=\frac{m^2}{q},$$
$$d=\frac{m^5+2b^2m^2q^2-3bfmq^3+f^2q^4}{m^2q(bm-fq)}, \ n=2m, \ p=\frac{2m^2}{q}, \ r=\frac{m^3}{q^2}, \ s=0; \qquad (4z4)$$

$$b=fs/k, d=cs/k, g=as/k, l=-k, m=(2k^2-s^2)/s, r=-k^2/s,$$
$$n=(k^2-2s^2)/s, p=k(k^2-2s^2)/s^2, q=(2k^2-s^2)/k; \qquad (4z5)$$

$$b=\frac{s(k^4-agkrs+k^2rs+a^2rs^2)}{k^3(gk-as)}, \ m=\frac{2k^4+rs^3}{k^2s}, \ n=\frac{k^4+2rs^3}{k^2s},$$
$$c=-\frac{g^2k^6-3agk^5s+2a^2k^4s^2+k^4s^3-agkrs^4+k^2rs^4+a^2rs^5}{k^3s^2(gk-as)},$$
$$d=\frac{agk^4-a^2k^3s-k^3s^2+g^2krs^2-agrs^3-krs^3}{k^3(gk-as)}, \ p=\frac{k^4+2rs^3}{ks^2},$$
$$f=-\frac{k^4+g^2k^2r-3agkrs+k^2rs+2a^2rs^2}{k^2(gk-as)}, \ l=\frac{rs}{k}, \ q=\frac{2k^4+rs^3}{k^3}. \qquad (4z6)$$

V. Cubic systems (1) with the line at infinity of multiplicity five

Theorem 4. *In the class of cubic systems of the form (1) the maximal multiplicity of the line at infinity* $Z = 0$ *is 5. Modulo the transformations* $x \to y, y \to x, t \to -t$, *the line* $Z = 0$ *has the multiplicity five for (1) if and only if one of the following six sets of conditions holds:*

$$a=0,\ b=0,\ c=0,\ f=0,\ g=0,\ k=0,\ l=0,\ m=0,\ n=0,\ p=0,\ r=0,\ s=0,\ q\neq 0; \quad (5z1)$$

$$b=0,\ c=0,\ d=2a,\ f=0,\ k=0,\ l=0,\ m=0,\ n=0,\ p=0,\ q=0,\ r=0,\ s=a^2,\ a\neq 0; \quad (5z2)$$

$$b=-\frac{as}{k},\ c=\frac{a(k^2-s^2)}{ks},\ d=\frac{a(k^2-s^2)}{k^2},\ f=-a,\ g=\frac{as}{k},\ m=\frac{2k^2-s^2}{s},\ l=-k,\ n=\frac{k^2-2s^2}{s},\ p=\frac{k(k^2-2s^2)}{s^2},\ q=\frac{2k^2-s^2}{k},\ r=-\frac{k^2}{s}; \quad (5z3)$$

$$a=0,\ b=-\frac{gk^2}{s^2},\ c=-\frac{2gk^2}{s^2},\ d=0,\ f=\frac{-2gk^3}{s^3},\ l=\frac{k^3}{s^2},\ m=\frac{3k^2}{s},\ n=\frac{3k^2}{s},\ p=\frac{3k^3}{s^2},\ q=3k,\ r=\frac{k^4}{s^3},\ g^2k^2-k^2s-s^3=0; \quad (5z4)$$

$$b=-\frac{as}{k},\ c=-\frac{2as}{k},\ d=2a,\ f=-\frac{a(k^2+2s^2)}{s^2},\ g=\frac{k^2+a^2s}{ak},\ l=\frac{k^3}{s^2},\ m=\frac{3k^2}{s},\ n=\frac{3k^2}{s},\ p=\frac{3k^3}{s^2},\ q=3k,\ r=\frac{k^4}{s^3},\ k^4-a^2k^2s-a^2s^3=0; \quad (5z5)$$

$$b=-\frac{k(-agk+k^2+a^2s+s^2)}{s(gk-as)},\ c=-\frac{2k(gk-2as)}{s^2},\ d=2a,\ f=-\frac{k^2(2gk-3as)}{s^3},\ l=\frac{k^3}{s^2},\ m=\frac{3k^2}{s},\ n=\frac{3k^2}{s},\ p=\frac{3k^3}{s^2},\ q=3k,\ r=\frac{k^4}{s^3},\ g^2k^2-2agks-k^2s+a^2s^2-s^3=0. \quad (5z6)$$

Theorems 3 and 4 from Sections IV and V can be proved similarly as Theorem 2.

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